The Many Meanings of Polarized Proof Theory

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Abstract

Proof theory is the study of why a proposition is true in a system, as opposed to the study of which propositions are true. In ordinary classical logic all proofs of the same proposition are indistinguishable, so its proof theory is essentially nonexistent. Polarized logic aims to fill this void as a rich proof theory with classical reasoning principles. In this report we examine polarized logic from the perspective of Noam Zeilberger's judgmental meaning-theories, and we apply this intuition to two different perspectives on polarization by Girard and Laurent.

1 Introduction

Mathematical logic is inherently concerned about what assertions, or judgments, are considered true in a particular logical system. Proof theory, on the other hands, studies *why* a particular judgment is true. Some systems, including intuitionistic logic and linear logic, have rich, constructive, and well-organized proof theories. Others, including classical logic, have only the bare minimum. And yet, classical logic lies at the foundation of logical reasoning principles. Without proof theory, what is the *meaning* of assertions in classical logic?

This question can be answered in two ways. The first approach, polarization, was developed by Girard (1991) as a logic equivalent in provability to classical logic, but which has a well-organized proof theory. Girard's LK logic has had applications in domains as diverse as game semantics (Laurent, 2002; Laurent and Regnier, 2003), proof search (Liang and Miller, 2009; McLaughlin and Pfenning, 2009), and evaluation order of programming languages (Levy, 2003; Zeilberger, 2009). It also has interesting connections to linear logic (Laurent, 2002) and focusing Andreoli (1992), and it explains the "meaning" of classical logic by considering CPS-like translations.

Another answer to the question takes a more literal and intuitive approach to *meaning*. Drawing on a technique of Dummett (1991), Zeilberger (2008) proposed two ways of attributing meaning to classical propositions themselves.

The first perspective, called verificationist, takes the meaning of a proposition to be how it is constructed, via its *canonical proofs*. In the case of the conjunction operator $\wedge$, the verificationist perspective leads to the following two (intuitionistic) rules:

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$$

The second perspective, called pragmatist, says that a proposition is defined by how it is used. This perspective leads to a logically equivalent set of rules which has a slightly different structure:

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \qquad \frac{\Gamma \vdash A \wedge B \qquad \Gamma, A, B \vdash C}{\Gamma \vdash C}$$

Alternatively, it is possible to think of the pragmatist perspective as being defined by the canonical *refutations* of a proposition, as opposed to its canonical proofs.

Zeilberger shows that these two perspectives on the meaning of classical logic are exactly the refinement polarized logic makes by splitting propositions into positive and negative fragments. In this paper we examine the field of polarized logic starting from Zeilberger's judgmental interpretation in Section 3. We compare this presentation to Girard's original formulation of polarization (1991) in Section 4, and also to Laurent and Regnier's presentation in terms of linear logic (2003) in Section 5. We conclude with a short discussion of related work in Section 6.

But first, we introduce the preliminaries of well-organized proof theory and the problems with classical logic in Section 2.

2 Preliminaries: Proof Theory

What do Girard (1991) and others mean when they talk about a constructive or well-organized proof theory? The most essential component of a proof theory is the principle of *cut-elimination*. Informally, the cut rule is just the rule of modus ponens: if a proposition A is true and the proposition A implies B, then B is also true. Cut-elimination holds for a logic if every proof containing a cut rule can be replaced by a cut-free proof.

Cut-elimination defines an equivalence relation on proofs, the smallest congruence relation containing the following rule: if a derivation $\mathcal{D}$ is replaced by a derivation $\mathcal{D}'$ in the cut-elimination algorithm, then $\mathcal{D} = \mathcal{D}'$.

Classical logic does satisfy cut elimination (Gentzen, 1935), but there are still other properties we might want in a constructive proof theory. Using intuitionistic logic and linear logic as examples, we can consider some of their features.

- The word *constructive* often recalls the disjunction property of intuitionistic logic, which states that a cut-free proof of $A \vee B$ is either a proof of

A or a proof of B. However, linear logic is considered constructive, but one of its disjunctions, $\parr$, does not satisfy the disjunction property. So the disjunction property does not seem to be an appropriate definition of a constructive logic.

- Both intuitionistic and linear logics have computational interpretations via the Curry-Howard correspondence. However, Parigot (1992) and later Curien and Herbelin (2000) introduced Curry-Howard interpretations for classical logic, so simply having a computational interpretation is not sufficient for a proof theory.

- Intuitionistic and linear logics also both have interesting denotational semantics which must, necessarily, respect the cut-elimination relation on proofs. For intuitionistic logic we can consider Hyland and Ong's fully abstract game-theoretic model for PCF (2000), and for linear logic we can consider Girard's semantics based on coherence spaces (1987). However, the only semantics of classical logic are *degenerate* ones, in which every proof of the same judgment is mapped to the same element in the semantics.

Girard claims that it is the presence of a non-degenerate denotational semantics that defines a well-organized proof theory. In the rest of the section, we will finally introduce classical logic and show why all of its denotational semantics are degenerate.

2.1 Classical logic

We start off by briefly reviewing Gentzen's 1935 formulation of classical logic, called LK. The types of LK include the following: atomic propositions X, truth $\mathbf{T}$, and falsity $\mathbf{F}$; the unary negation operator $\neg$; and as the binary connectives and $\wedge$, or $\vee$. We denote propositions with the meta-variable A.

$$A ::= X \mid X^{\perp} \mid \mathbf{T} \mid \mathbf{F} \mid \neg A \mid A \wedge B \mid A \vee B$$

Each proposition has a De Morgan dual, written as a meta-operation $A^{\perp}$.

Figure 1 shows a one-sided sequent calculus formulation of LK. We use explicit structural rules akin to those of linear logic.

Cut-elimination is a property that states that any proof in LK can be replaced by an equivalent cut-free proof. It relies primarily on the admissibility of the cut rule, due to Gentzen (1935).

Lemma 1 (Admissibility of Cut)**.** *Given a cut-free proof $\mathcal{D}_1$ of $\vdash \Delta_1, A$ and a cut-free proof $\mathcal{D}_2$ of $\vdash A^{\perp}, \Delta_2$, there is a cut-free proof of $\vdash \Delta_1, \Delta_2$.*

Proof Sketch. By induction primarily on A, and secondarily on the structures of $\mathcal{D}_1$ and $\mathcal{D}_2$ simultaneously. To get a feeling for the cut-elimination procedure, we will sketch a few cases.

$$\frac{}{\vdash A^{\perp}, A}\ \textsc{axiom} \qquad \frac{\vdash \Delta_1, A \quad \vdash A^{\perp}, \Delta_2}{\vdash \Delta_1, \Delta_2}\ \textsc{cut}$$

$$\frac{}{\vdash \mathbf{T}}\ \mathbf{T} \qquad \frac{\vdash \Delta, A^{\perp}}{\vdash \Delta, \neg A}\ \neg$$

$$\frac{\vdash \Delta_1, A_1 \quad \vdash \Delta_2, A_2}{\vdash \Delta_1, \Delta_2, A_1 \wedge A_2}\ \wedge \qquad \frac{\vdash \Delta, A_i}{\vdash \Delta, A_1 \vee A_2}\ \vee$$

$$\frac{\vdash \Delta}{\vdash \Delta, A}\ \textsc{w} \qquad \frac{\vdash \Delta, A, A}{\vdash \Delta, A}\ \textsc{c}$$

Figure 1: Sequent calculus presentation of LK

For example, suppose $\mathcal{D}_1$ is an introduction rule of $A = A_1 \wedge A_2$ and $\mathcal{D}_2$ is an introduction rule for $A^{\perp} = {A_1}^{\perp} \vee {A_2}^{\perp}$, as follows:

$$\mathcal{D}_1 = \frac{\vdash \Delta_{11}, A_1 \qquad \vdash \Delta_{12}, A_2}{\vdash \Delta_{11}, \Delta_{12}, A_1 \wedge A_2} \qquad \mathcal{D}_2 = \frac{\vdash {A_1}^{\perp}, {A_2}^{\perp}, \Delta_2}{\vdash {A_1}^{\perp} \vee {A_2}^{\perp}, \Delta_2}$$

Then a cut-free proof of $\vdash \Delta_{11}, \Delta_{12}, \Delta_2$ can be constructed from the inductive hypotheses:

$$\frac{\vdash \Delta_{11}, A_1 \qquad \dfrac{\vdash \Delta_{12}, A_2 \qquad \vdash {A_1}^{\perp}, {A_2}^{\perp}, \Delta_2}{\vdash {A_1}^{\perp}, \Delta_{12}, \Delta_2}}{\vdash \Delta_{11}, \Delta_{12}, \Delta_2}$$

On the other hand, suppose $\mathcal{D}_1$ concludes with a weakening rule on A:

$$\mathcal{D}_1 = \frac{\vdash \Delta_1}{\vdash \Delta_1, A}$$

Then a cut-free proof of $\vdash \Delta_1, \Delta_2$ could be constructed by applying the weakening rule for every proposition in Δ_2. $\square$

2.2 The problem with classical logic

An example by Lafont (Girard et al., 1989, Appendix B) illustrates that all derivations of the same judgment in LK are equivalent via cut-elimination. Such a logic can have only degenerate denotational semantics—boolean algebras—and so we say the proof theory is also degenerate.

Consider any two derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of $\vdash \Delta$. By weakening these can be transformed into derivations of $\vdash \Delta, A$ and $\Delta, A^{\perp}$, respectively for any proposition A. Applying the cut rule followed immediately by contractions on Δ, we obtain a new proof $\mathcal{D}_0$ of $\vdash \Delta$.

$$\mathcal{D}_0 = \dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\vdash \Delta}}{\vdash \Delta, A}\,\textsc{w} \qquad \dfrac{\dfrac{\mathcal{D}_2}{\vdash \Delta}}{\vdash \Delta, A^{\perp}}\,\textsc{w}}{\vdash \Delta, \Delta}\,\textsc{cut}}{\vdash \Delta}\,\textsc{c}$$

By the cut-elimination procedure outlined above, this proof is of equivalent to

$$\mathcal{D}_0 = \dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\vdash \Delta}}{\vdash \Delta, \Delta}\,\textsc{w}}{\vdash \Delta}\,\textsc{c}$$

which must be equal to $\mathcal{D}_1$ itself. However, by the same reasoning we have $\mathcal{D}_0$ equivalent to $\mathcal{D}_2$; thus $\mathcal{D}_1$ and $\mathcal{D}_2$ are necessarily equivalent.

Girard's search for a semantics of LK is one in which not all proofs are identified. The approach of polarization solves this problem by using information implicit in a judgment to convey how proofs of that judgment are constructed. That is, the judgment has *meaning* other than just its provability. Complementing the meaning of a judgment is the meaning of the propositions in the judgment. In the next section we will explore different notions of meaning and what they convey about the shape of a proof.

3 The Meanings of the Connectives

The idea of *justifying* the laws of logic comes from a series of lectures by Dummett (1991) in which he argues that standard presentations of logic can be understood by attributing meaning to the connectives in various ways. Dummett proposes that the meaning of a proposition might be defined either by its *canonical proofs* or by its *canonical refutations.*

The so-called *verificationist* meaning-theory states that the meaning of a proposition is its definition as given by a *canonical proof.* Essentially, a proposition is defined by how it is constructed. In the Curry-Howard isomorphism, these canonical proofs correspond to values.

On the other hand, the *pragmatist* meaning-theory states that the meaning of a proposition is a *canonical refutation* of that proposition. That is, a proposition is defined by what things it can be used to prove.

Zeilberger (2008) makes Dummett's justifications of the logical laws explicit by integrating them with the judgmental method as introduced by Martin-Löf (1996) and Pfenning and Davies (2001). The resulting logic, which we call ZU[1], judgmentally distinguishes propositions proved via the verificationist

$$\text{Formulas} \qquad A ::= X \mid \mathbf{T} \mid \mathbf{F} \mid A \wedge B \mid A \vee B \mid \neg A$$

$$\frac{}{X\,\mathbf{triv} \Rightarrow X\,\mathbf{triv}} \qquad \frac{}{X\,\mathbf{absurd} \Rightarrow X\,\mathbf{absurd}}$$

$$\frac{}{A\,\mathbf{false} \Rightarrow \neg A\,\mathbf{triv}} \qquad \frac{}{A\,\mathbf{true} \Rightarrow \neg A\,\mathbf{absurd}}$$

$$\frac{}{\cdot \Rightarrow \mathbf{T}\,\mathbf{triv}} \qquad \frac{}{\cdot \Rightarrow \mathbf{F}\,\mathbf{absurd}}$$

$$\frac{\Delta_1 \Rightarrow A_1\,\mathbf{triv} \quad \Delta_2 \Rightarrow A_2\,\mathbf{triv}}{\Delta_1, \Delta_2 \Rightarrow A_1 \wedge A_2\,\mathbf{triv}} \qquad \frac{\Delta \Rightarrow A_i\,\mathbf{absurd}}{\Delta \Rightarrow A_1 \wedge A_2\,\mathbf{absurd}}$$

$$\frac{\Delta \Rightarrow A_i\,\mathbf{triv}}{\Delta \Rightarrow A_1 \vee A_2\,\mathbf{triv}} \qquad \frac{\Delta_1 \Rightarrow A_1\,\mathbf{absurd} \quad \Delta_2 \Rightarrow A_2\,\mathbf{absurd}}{\Delta_1, \Delta_2 \Rightarrow A_1 \vee A_2\,\mathbf{absurd}}$$

Figure 2: Direct proofs and refutations

meaning-theory from propositions proved via the pragmatist meaning-theory. Verificationist judgments have the form $\Delta \Rightarrow A\,\mathbf{triv}$, which means that A is "trivially" true from Δ. Pragmatist judgments have the form $\Delta \Rightarrow A\,\mathbf{absurd}$, which means that A is directly "absurd" from Δ. We give a presentation of these judgments in Figure 2.[2]

However, *direct* proofs and refutations are not enough to express all reasonable classical proofs. For this we introduce two new judgments, $A\,\mathbf{false}$ and $A\,\mathbf{true}$, which state, respectively, that there is a contradiction of $A\,\mathbf{triv}$ and $A\,\mathbf{false}$ respectively. Thus a proof of $A\,\mathbf{false}$ is an instance of intuitionistic proof of negation—assume $A\,\mathbf{triv}$ and prove a contradiction. On the other hand, a proof of $A\,\mathbf{true}$ is an instance of classical proof by contradiction—assume $A\,\mathbf{absurd}$ and derive a contradiction.

$$\frac{\forall(\Delta \Rightarrow A\,\mathbf{triv}): \quad \Gamma, \Delta \vdash \mathbf{contra}}{\Gamma \vdash A\,\mathbf{false}} \qquad \frac{\forall(\Delta \Rightarrow A\,\mathbf{absurd}): \quad \Gamma, \Delta \vdash \mathbf{contra}}{\Gamma \vdash A\,\mathbf{true}}$$

A contradiction simply means that there is some proposition that is both trivial and false, or both true and absurd.

$$\frac{\Gamma_1 \vdash A\,\mathbf{triv} \quad \Gamma_2 = A\,\mathbf{false}}{\Gamma_1, \Gamma_2 \vdash \mathbf{contra}} \qquad \frac{\Gamma_1 = A\,\mathbf{true} \quad \Gamma_2 \vdash A\,\mathbf{absurd}}{\Gamma_1, \Gamma_2 \vdash \mathbf{contra}}$$

[1]Zeilberger Unity. Zeilberger (2008) does not name the logic presented in his paper, but he calls the associated calculus CU for the Calculus of Unity, in reference to Girard's Logic of Unity (1993) (LU). Girard's system is quite different from ZU, but has similar goals: to combine classical, intuitionistic, and linear logics in a single system.

[2]In Zeilberger's presentation, propositions are partitioned into positive and negative fragments, and distinguished using syntax akin to linear logic. Positive propositions are used in trivial and false judgments, and negative propositions are used in absurd and true judgments. We will revisit this notation later in this paper, but for now stick with unpolarized propositions.

$$\frac{}{\cdot \vdash \cdot} \qquad \frac{\Gamma_1 \vdash X\,\mathbf{triv} \quad \Gamma_2 \vdash \Delta}{\Gamma_1, \Gamma_2 \vdash X\,\mathbf{triv}, \Delta} \qquad \frac{\Gamma_1 \vdash A\,\mathbf{false} \quad \Gamma_2 \vdash \Delta}{\Gamma_1, \Gamma_2 \vdash A\,\mathbf{false}, \Delta}$$

$$\frac{\Gamma_1 \vdash A\,\mathbf{true} \quad \Gamma_2 \vdash \Delta}{\Gamma_1, \Gamma_2 \vdash A\,\mathbf{true}, \Delta} \qquad \frac{\Gamma_1 \vdash A\,\mathbf{false} \quad \Gamma_2 \vdash \Delta}{\Gamma_1, \Gamma_2 \vdash A\,\mathbf{false}, \Delta}$$

Figure 3: Context judgment

Despite the awkward presentation of this rule, Zeilberger prove the more general form as well: that if $\Gamma_1 \vdash A\,\mathbf{triv}$ and $\Gamma_2 \vdash A\,\mathbf{false}$ then $\Gamma_1, \Gamma_2 \vdash \mathbf{contra}$.

Finally, we use the notation $\Gamma \vdash A\,\mathbf{triv}$ to mean that there is a direct proof $\Delta \Rightarrow A\,\mathbf{triv}$ requiring hypotheses Δ, and that Γ is sufficient to prove every element of Δ. A similar justification exists for $\Gamma \vdash A\,\mathbf{absurd}$. The judgment $\Gamma \vdash \Delta$ is defined in Figure 3.

$$\frac{\Delta \Rightarrow A\,\mathbf{triv} \quad \Gamma \vdash \Delta}{\Gamma \vdash A\,\mathbf{triv}} \qquad \frac{\Delta \Rightarrow A\,\mathbf{absurd} \quad \Gamma \vdash \Delta}{\Gamma \vdash A\,\mathbf{absurd}}$$

This paper's presentation of ZU uses explicit linearity, which means that all hypotheses are used exactly once, but that they can be explicitly weakened and contracted:

$$\frac{\Gamma \vdash J'}{\Gamma, J \vdash J'} \qquad \frac{\Gamma, J, J \vdash J'}{\Gamma, J \vdash J'}$$

This is in contrast to Zeilberger's presentation, which mixes explicit and implicit uses of linearity. In our setting, the explicit linearity throughout makes it easier to understand the difference between direct and indirect proofs.

3.1 Modular extensions of ZU

The rules for the judgments $\Gamma \vdash J$ are in some sense the *meaning* of those judgments; they should not change under extensions of the logic. The meaning of the *connectives*, on the other hand, does change under extensions, but these are given exclusively by the judgments $\Delta \Rightarrow J$. To add a new connective, therefore, it suffices to give its definition in terms of $\Delta \Rightarrow A\,\mathbf{triv}$ and $\Gamma \Rightarrow A\,\mathbf{absurd}$; the meanings of $A\,\mathbf{false}$ and $A\,\mathbf{true}$ are thus automatically inferred.

In this section we add implication to obtain what is known as full propositional classical logic. Direct proofs of $A \to B$ will just be direct proofs of $\neg\,(A \wedge \neg\,B)$, while direct refutations will be direct refutations of $\neg\,A \vee B$.

$$\frac{}{A \wedge \neg\,B\,\mathbf{false} \Rightarrow A \to B\,\mathbf{triv}} \qquad \frac{\Delta \Rightarrow B\,\mathbf{absurd}}{A\,\mathbf{true}, \Delta \Rightarrow A \to B\,\mathbf{absurd}}$$

Notice that since negation does not distribute isomorphically with $\wedge$ and $\vee$, the two interpretations are not equivalent in LC, but they give two different *meanings* to the implication operator.

3.2 Fragments of ZU

The judgmental presentation makes it clear that there are two different kinds of "truthfulness" and two different kinds of "falsity," but it is not yet obvious what is the relationship between the two.

Zeilberger points out that intuitionistic logic is a special case of ZU in which the judgments are restricted to $A\,\mathbf{triv}$, $A\,\mathbf{false}$, and $\mathbf{contra}$. We call this the *positive* fragment. In it, negation introduction is allowable by the *false* rule, but double negation elimination (i.e. the *true* rule) is not. Zeilberger proves that a judgment J holds if $|J|$ holds intuitionistically, where

$$|A\,\mathbf{triv}| = A \qquad |A\,\mathbf{false}| = \neg\, A \qquad |\mathbf{contra}| = \mathbf{F}.$$

On the other hand, the *negative* fragment consists of the judgments $A\,\mathbf{true}$, $A\,\mathbf{absurd}$, and $\mathbf{contra}$. This fragment allows double negation elimination by *true* judgment, but does not allow for the usual negation introduction. This fragment has been independently studied: Urbas' dual intuitionistic logic (1996).

As the names suggest, the positive and negative fragments are in duality with each other. Zeilberger defines a meta-operation $J^{\perp}$ on judgments as follows:

$$\begin{array}{cc} (A\,\mathbf{triv})^{\perp} = A^{\perp}\,\mathbf{absurd} & (A\,\mathbf{false})^{\perp} = A^{\perp}\,\mathbf{true} \\ (A\,\mathbf{true})^{\perp} = A^{\perp}\,\mathbf{false} & (A\,\mathbf{absurd})^{\perp} = A^{\perp}\,\mathbf{triv} \end{array}$$

where $A^{\perp}$ is defined as expected by De Morgan duality. It is easy to see that $\Gamma \vdash J$ if and only if $\Gamma^{\perp} \vdash J^{\perp}$.

In order to obtain *classical* reasoning principles in ZU, we must surely include both the positive and the negative fragments. But even further, there must be a way to unify proofs by contradiction of the type $A\,\mathbf{true}$ with direct proofs $A\,\mathbf{triv}$, and vice versa. We do this by adding *shift rules* that embed indirect proofs into direct proofs, and indirect refutations into direct refutations, as follows:

$$\frac{}{A\,\mathbf{true} \Rightarrow\, \downarrow A\,\mathbf{triv}}\ \downarrow \qquad \frac{}{A\,\mathbf{false} \Rightarrow\, \uparrow A\,\mathbf{absurd}}\ \uparrow$$

The shift operators conclude the definition of ZU. The relationships between negation, duality, and the shift operators can be summarized nicely in Figure 4, reproduced from Zeilberger (2008).

3.3 Derived rules

In this section we will derive inference rules $\Gamma \vdash J$ for each connective, to better understand what the other judgments *false* and *true* mean in each case. To get a handle on what proofs in ZU actually look like in practice, we can derive some rules for the *false* and *true* judgments that look more familiar. In the process we will use the following properties from Zeilberger (2008):

- (Identity) If $J \in \Gamma$ then $\Gamma \vdash J$.
- (Context Identity) If $\Delta \Rightarrow J$ then $\Delta \vdash J$.

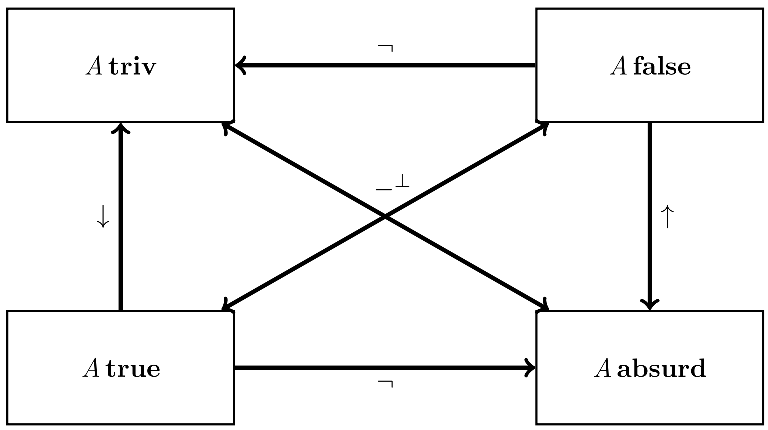

Figure 4: Relationship between judgments (Zeilberger, 2008)

- (Reduction$^+$) If $\Gamma_1 \vdash A\,\textbf{triv}$ and $\Gamma_2 \vdash A\,\textbf{false}$ then $\Gamma_1, \Gamma_2 \vdash \textbf{contra}$.
- (Reduction$^-$) If $\Gamma_1 \vdash A\,\textbf{true}$ and $\Gamma_2 \vdash A\,\textbf{absurd}$ then $\Gamma_1, \Gamma_2 \vdash \textbf{contra}$.
- (Substitution) If $\Gamma_1 \vdash J_1$ and $\Gamma_2, J_1 \vdash J_2$ then $\Gamma_1, \Gamma_2 \vdash J_2$.

Atomic propositions, F and T. By examination there is no context Δ such that $\Delta \Rightarrow \textbf{F}\,\textbf{triv}$; therefore $\Gamma \vdash \textbf{F}\,\textbf{false}$ is vacuously true for any Γ. On the other hand, $\cdot \Rightarrow \textbf{F}\,\textbf{absurd}$ so we have $\Gamma \vdash \textbf{F}\,\textbf{true}$ as long as $\Gamma \vdash \textbf{contra}$. We can thus generate the following rules for the four inferred rules for **F**:

$$\text{(no rule for } \textit{triv}\text{)} \qquad \frac{}{\Gamma \vdash \textbf{F}\,\textbf{false}} \qquad \frac{\Gamma \vdash \textbf{contra}}{\Gamma \vdash \textbf{F}\,\textbf{true}} \qquad \frac{}{\cdot \vdash \textbf{F}\,\textbf{absurd}}$$

By duality, we can immediate generate an equivalent set of rules for **T**:

$$\frac{}{\cdot \vdash \textbf{T}\,\textbf{triv}} \qquad \frac{\Gamma \vdash \textbf{contra}}{\Gamma \vdash \textbf{T}\,\textbf{false}} \qquad \frac{}{\Gamma \vdash \textbf{T}\,\textbf{true}} \qquad \text{(no rule for } \textit{absurd}\text{)}$$

Binary operators, $\wedge$ and $\vee$. We start by asking, when does $\Gamma \vdash A_1 \wedge A_2\,\textbf{false}$ hold? Notice that $\Delta \Rightarrow A_1 \wedge A_2\,\textbf{triv}$ only if $\Delta = \Delta_1, \Delta_2$ such that $\Delta_1 \Rightarrow A_1\,\textbf{triv}$ and $\Delta_2 \Rightarrow A_2\,\textbf{triv}$. Then to show $\Gamma, \Delta \vdash \textbf{contra}$, it suffices to show that $\Gamma, A_1\,\textbf{triv}, A_2\,\textbf{triv} \vdash \textbf{contra}$. Thus we have the following derived rule:

$$\frac{\Gamma, A_1\,\textbf{triv}, A_2\,\textbf{triv} \vdash \textbf{contra}}{\Gamma \vdash A_1 \wedge A_2\,\textbf{false}}$$

Compared to the *absurd* rule, the *false* rule has more flexibility in how its resources are used:

$$\frac{\Gamma \vdash A_i\,\textbf{absurd}}{\Gamma \vdash A_1 \wedge A_2\,\textbf{absurd}}$$

By duality, these rules mirror the "truth" rules for $\vee$:

$$\frac{\Gamma \vdash A_i \,\mathbf{triv}}{\Gamma \vdash A_1 \vee A_2 \,\mathbf{triv}} \qquad \frac{\Gamma, A_1 \,\mathbf{absurd}, A_2 \,\mathbf{absurd} \vdash \mathbf{contra}}{\Gamma \vdash A_1 \vee A_2 \,\mathbf{true}}$$

Next, to conclude $\Gamma \vdash A_1 \wedge A_2 \,\mathbf{true}$ we must prove $\Gamma, \Delta \vdash \mathbf{contra}$ whenever $\Delta \Rightarrow A_1 \,\mathbf{absurd}$ or $\Delta \Rightarrow A_2 \,\mathbf{absurd}$. If both $\Gamma \vdash A_1 \,\mathbf{true}$ and $\Gamma \vdash A_2 \,\mathbf{true}$ hold (for the same Γ), then the result is assured. Thus we have the following derived rule:

$$\frac{\Gamma \vdash A_1 \,\mathbf{true} \quad \Gamma \vdash A_2 \,\mathbf{true}}{\Gamma \vdash A_1 \wedge A_2 \,\mathbf{true}}$$

Notice how this rule differs from the rule for $A_1 \wedge A_2 \,\mathbf{triv}$:

$$\frac{\Gamma_1 \vdash A_1 \,\mathbf{triv} \quad \Gamma_2 \vdash A_2 \,\mathbf{triv}}{\Gamma_1, \Gamma_2 \vdash A_1 \wedge A_2 \,\mathbf{triv}}$$

The former uses the same context, Γ, in both hypotheses, while the later uses different contexts and concatenates them. By weakening and contraction these presentations are equivalent in provability, but again the judgmental method highlights how hypotheses are actually used in each proof.

By duality we complete the picture for the connective $\vee$:

$$\frac{\Gamma \vdash A_1 \,\mathbf{false} \quad \Gamma \vdash A_2 \,\mathbf{false}}{\Gamma \vdash A_1 \vee A_2 \,\mathbf{false}} \qquad \frac{\Gamma_1 \vdash A_1 \,\mathbf{absurd} \quad \Gamma_2 \vdash A_2 \,\mathbf{absurd}}{\Gamma_1, \Gamma_2 \vdash A_1 \vee A_2 \,\mathbf{absurd}}$$

Negation, $\neg$. A direct proof of $\neg A$ is simply an indirect refutation of A, and vice versa for direct refutations:

$$\frac{\Gamma \vdash A \,\mathbf{false}}{\Gamma \vdash \neg A \,\mathbf{triv}} \qquad \frac{\Gamma \vdash A \,\mathbf{true}}{\Gamma \vdash \neg A \,\mathbf{absurd}}$$

The indirect judgments say that assuming the hypothesis leads to a contradiction.

$$\frac{\Gamma, A \,\mathbf{false} \vdash \mathbf{contra}}{\Gamma \vdash \neg A \,\mathbf{false}} \qquad \frac{\Gamma, A \,\mathbf{true} \vdash \mathbf{contra}}{\Gamma \vdash \neg A \,\mathbf{true}}$$

Implication, $\rightarrow$. The direct proofs and refutations we defined for $\rightarrow$ correspond to two different interpretations of implication in classical logic. The positive interpretation corresponds to $\neg(A \wedge \neg B)$, while the negative interpretation corresponds to $\neg A \vee B$. Though the direct constructions look a little mysterious, we can recover the usual interpretations of implication through inferred rules.

$$\frac{\Gamma, A \,\mathbf{triv} \vdash B \,\mathbf{triv}}{\Gamma \vdash A \rightarrow B \,\mathbf{triv}} \qquad \frac{\Gamma_1 \vdash A \,\mathbf{triv} \quad \Gamma_2 \vdash B \,\mathbf{false}}{\Gamma_1, \Gamma_2 \vdash A \rightarrow B \,\mathbf{false}}$$

The inferred *absurd* rule has a very similar structure to the *false* rule, but the indirect proof of $A \rightarrow B \,\mathbf{true}$ recalls the dual intuitionistic proof strategy intrinsic to the negative fragment.

$$\frac{\Gamma, B \,\mathbf{absurd} \vdash A \,\mathbf{absurd}}{\Gamma \vdash A \rightarrow B \,\mathbf{true}} \qquad \frac{\Gamma_1 \vdash A \,\mathbf{true} \quad \Gamma_2 \vdash B \,\mathbf{absurd}}{\Gamma_1, \Gamma_2 \vdash A \rightarrow B \,\mathbf{absurd}}$$

Shift operators, $\downarrow$ and $\uparrow$. A direct proof of $\downarrow A\,\mathbf{triv}$ is clearly just an indirect proof of $A\,\mathbf{true}$, but what is an indirect refutation of $\downarrow A\,\mathbf{false}$? It is easy to derive that the judgment holds provided $A\,\mathbf{true}$ leads to a contradiction.

$$\frac{\Gamma \vdash A\,\mathbf{true}}{\Gamma \vdash \downarrow A\,\mathbf{triv}} \qquad \frac{\Gamma, A\,\mathbf{true} \vdash \mathbf{contra}}{\Gamma \vdash \downarrow A\,\mathbf{false}}$$

Notice that this is equivalent to saying $\neg A\,\mathbf{true}$. By duality, $\uparrow A\,\mathbf{true}$ is equivalent to $\neg A\,\mathbf{false}$.

As a consequence, Zeilberger (2008) shows that we also have $\neg\neg A\,\mathbf{triv} \equiv \downarrow\uparrow A\,\mathbf{triv}$ and $\neg\neg A\,\mathbf{absurd} \equiv \uparrow\downarrow A\,\mathbf{absurd}$.

4 Introducing Polarity

In the previous section we saw how the structure of direct proofs of the form $A\,\mathbf{triv}$ differs from the structure of indirect proofs of the form $A\,\mathbf{true}$. Girard (1991) proposes another way to distinguish these classes of proofs: from the structure of the proposition itself.

Girard (1991)'s work starts by partitioning the propositions of classical logic into two parts: positive and negative. Intuitively, positive propositions have direct proofs and negative propositions have indirect proofs. We will be working in a one-sided sequent calculus, so refutations of A are subsumed by proofs of $A^{\perp}$.

In the following section, we denote the dual of an atomic type X as $\overline{X}$ and extend this syntax to $\mathbf{T}$ and $\mathbf{F}$. That is, the dual of $\mathbf{T}$ is written $\overline{\mathbf{T}}$ and is distinguished from $\mathbf{F}$.

The polarity of a formula is computed by the following algorithm.

1. Atomic formulas X, $\mathbf{T}$, and $\mathbf{F}$ are positive;
2. The duals of atomic formulas, $\overline{X}$, $\overline{\mathbf{T}}$, and $\overline{\mathbf{F}}$, are negative;
3. The polarity of compound formulas are dictated by the following chart:

A	B	$A \wedge B$	$A \vee B$	$\neg A$
$+$	$+$	$+$	$+$	$+$
$-$	$+$	$+$	$-$	$-$
$+$	$-$	$+$	$-$	
$-$	$-$	$-$	$-$	

We use P to refer to a positive proposition, and N to a negative proposition. An alternative presentation of the positive/negative distinction is axiomatic, as follows:

$$\begin{array}{llllllllll} P & := X & \mid \mathbf{T} & \mid \mathbf{F} & \mid P_1 \wedge P_2 & \mid N \wedge P & \mid P \wedge N & \mid P_1 \vee P_2 & \mid \neg P \\ N & := \overline{X} & \mid \overline{\mathbf{T}} & \mid \overline{\mathbf{F}} & \mid N_1 \wedge N_2 & \mid N \vee P & \mid P \vee N & \mid N_1 \vee N_2 & \mid \neg N \end{array}$$

Notice that negation does not change the polarity of a formula, although De Morgan dualization does.

$$\frac{}{\vdash P^\perp ; P}\ \textsc{axiom} \qquad \frac{\vdash \Delta ; P}{\vdash \Delta, P ; \cdot}\ \textsc{dereliction}$$

$$\frac{\vdash \Delta_1 ; P \quad \vdash P^\perp, \Delta_2 ; \Theta}{\vdash \Delta_1, \Delta_2 ; \Theta}\ \textsc{cut-P} \qquad \frac{\vdash \Delta_1, N ; \cdot \quad \vdash N^\perp, \Delta_2 ; \Theta}{\vdash \Delta_1, \Delta_2 ; \Theta}\ \textsc{cut-N}$$

$$\frac{\vdash \Delta ; \Theta}{\vdash A, \Delta ; \Theta}\ \textsc{weakening} \qquad \frac{\vdash A, A, \Delta ; \Theta}{\vdash A, \Delta ; \Theta}\ \textsc{contraction}$$

Figure 5: LC (Girard, 1991)

4.1 LC

Girard (1991) introduces the sequent calculus LC in order to distinguish direct proofs from indirect proofs, since the distinction is not possible in LK directly.[3] The key contribution of LC is the *stoup* Θ, a context consisting of either zero or one positive propositions. Judgments have the form $\vdash \Delta ; \Theta$, and the presence of a non-empty stoup *focuses* the judgment on the positive proposition P. Judgments of this form correspond to direct proofs of $P\,\mathbf{triv}$ in Zeilberger's system. The context Δ may contain both positive and negative propositions. Although indirect proofs (of the form $A\,\mathbf{true}$) are also focused in Zeilberger's ZU, in LC negative propositions remain unfocused.

Other variations of these focused systems have been presented in the literature (Andreoli, 1992; Liang and Miller, 2009). The purpose of focusing is to restrict the ways to construct a proof so that, while sound and complete with respect to the unfocused system, the focused derivations collapse equivalent unfocused derivations to a more normal form.

The basic rules of LC are reproduced in Figure 5. The full presentation is rather unwieldy, due to the large number of inference rules for each connective. In fact, for each connective we need to provide distinct rules for each combination of the polarities.

Starting off with the units, there is exactly one direct proof of Θ, and an indirect proof of $\overline{\mathbf{F}}$ from any hypotheses. On the other hand, if we can prove one of Δ or Θ then we can add $\overline{\mathbf{T}}$ to the collection of indirect proofs.

$$\frac{}{\vdash \cdot ; \mathbf{T}} \qquad \frac{}{\vdash \Delta, \overline{\mathbf{F}} ; \Theta} \qquad \frac{\vdash \Delta ; \Theta}{\vdash \Delta, \overline{\mathbf{T}} ; \Theta}$$

For $\wedge$, a direct proof is constructed if at least one of its subpropositions has

[3] In fact, Girard framed this in terms of a semantics of correlation spaces, an algebraic structure based loosely on the coherence space denotational semantics of linear logic. In LC, indirect proofs correspond to cliques of correlation spaces, while direct proofs correspond to the special case of *central cliques*.

a direct proof:

$$\frac{\vdash \Delta_1; P_1 \quad \vdash \Delta_2; P_2}{\vdash \Delta_1, \Delta_2; P_1 \wedge P_2} \qquad \frac{\vdash \Delta_1; P \quad \vdash \Delta_2, N; \cdot}{\vdash \Delta_1, \Delta_2; P \wedge N} \qquad \frac{\vdash \Delta_1, N; \cdot \quad \vdash \Delta_2; P}{\vdash \Delta_1, \Delta_2; N \wedge P}$$

An indirect proof, on the other hand, must have indirect proofs of its subpropositions, but the rest of the conclusions can be duplicated, just as in the proofs of $A \wedge B$ **true**.

$$\frac{\vdash \Delta, N_1; \Theta \quad \vdash \Delta, N_2; \Theta}{\vdash \Delta, N_1 \wedge N_2; \Theta}$$

On the other hand, $A \vee B$ has a direct proof if either A or B does. If not, it suffices to give an indirect proof for *either* A or B, concurrently.

$$\frac{\vdash \Delta; P_i}{\vdash \Delta; P_1 \vee P_2} \qquad \frac{\vdash \Delta, P, N; \Theta}{\vdash \Delta, P \vee N; \Theta} \qquad \frac{\vdash \Delta, N, P; \Theta}{\vdash \Delta, N \vee P; \Theta} \qquad \frac{\vdash \Delta, N_1, N_2; \Theta}{\vdash \Delta, N_1 \vee N_2; \Theta}$$

Compare the negative disjunction rules to the corresponding rule in ZU:

$$\frac{\Gamma, A_1\, \mathbf{absurd}, A_2\, \mathbf{absurd} \vdash \mathbf{contra}}{\Gamma \vdash A_1 \vee A_2\, \mathbf{true}}$$

There are similar rules for negation. There exists a direct proof of $\neg A$ if there is an indirect proof of $A^{\perp}$. In addition, there is an indirect proof of $\neg A$ if there is an indirect proof of $A^{\perp}$.

$$\frac{\vdash \Delta, P^{\perp}; \cdot}{\vdash \Delta; \neg P} \qquad \frac{\vdash \Delta, N^{\perp}; \Theta}{\vdash \Delta, \neg N; \Theta}$$

Compare these to the rules for negation in ZU:

$$\frac{\Gamma \vdash A\, \mathbf{false}}{\Gamma \vdash \neg A\, \mathbf{triv}} \qquad \frac{\Gamma, A\, \mathbf{true} \vdash \mathbf{contra}}{\Gamma \vdash \neg A\, \mathbf{true}}$$

4.2 Dereliction and promotion in the stoup.

The dereliction rule in Figure 5 unfocuses a proposition, and we can achieve the opposite effect through a derived rule. Notice that it is always possible to replace a formula by an equivalent one of a particular polarity. We write $\downarrow A$ for $A \wedge \mathbf{T}$, which is positive and equivalent to A, and $\uparrow A$ for $A \vee \overline{\mathbf{T}}$, which is negative and equivalent to A. The derived rule for $\downarrow A$ is a kind of promotion for negative propositions, while the derived rule for $\uparrow A$ does not change the focus of the proposition.

$$\dfrac{\dfrac{\vdash \Delta, N; \cdot \qquad \dfrac{}{\vdash \cdot; \mathbf{T}}}{\vdash \Delta; N \wedge \mathbf{T}}}{\vdash \Delta; \downarrow N} \qquad \dfrac{\dfrac{\dfrac{\vdash \Delta, P; \Theta}{\vdash \Delta, P, \mathbf{T}^{\perp}; \Theta}}{\vdash \Delta, P \vee \mathbf{T}^{\perp}; \Theta}}{\vdash \Delta, \uparrow P; \Theta}$$

Indeed, these derived rules correspond to the ones in ZU of the same name. More precisely, the $\uparrow$ rule embeds indirect refutations into direct refutations (both unfocused in LC), and the $\downarrow$ rule embeds indirect proofs into direct proofs.

4.3 Semantics

Girard's goal in developing LC was to construct a (non-degenerate) denotational semantics for it, which we briefly sketch here. Inspired by the coherence space semantics for linear logic (Girard, 1987), the denotational semantics for LC is based on *correlation spaces*, which consist of a coherence space together with some additional clique-like structure. Derivations are interpreted as cliques between correlation spaces, and in particular derivations with a non-empty stoup are interpreted as *central* cliques, having additional structure. These central cliques distinguish between proofs that in LK would be equivalent.

Girard (1991) proves that the denotation is invariant under cut-elimination and is non-degenerate, meaning that there are two proofs of the same sequent in LC that have different interpretations under the semantics.

4.4 Translation into LC

Following Zeilberger (2008), we can make the relationship between ZU and LC formal with a sound and complete translation between the two logics.[4]

First, define maps $(-)^+$ and $(-)^-$ on ZU propositions; the result will be an LC proposition of the appropriate polarity.

$$
\begin{aligned}
(\mathbf{T})^+ &= \mathbf{T} & (\mathbf{T})^- &= \overline{\mathbf{F}} \\
(\mathbf{F})^+ &= \mathbf{F} & (\mathbf{F})^- &= \overline{\mathbf{T}} \\
(A \wedge B)^+ &= A^+ \wedge B^+ & (A \wedge B)^- &= A^- \wedge B^- \\
(A \vee B)^+ &= A^+ \vee B^+ & (A \vee B)^- &= A^- \vee B^- \\
(\downarrow A)^+ &= A^- \wedge \mathbf{T} & (\downarrow A)^- &= A^- \\
(\uparrow A)^+ &= A^+ & (\uparrow A)^- &= A^+ \vee \overline{\mathbf{T}} \\
(\neg\, A)^+ &= \neg\, A^+ & (\neg\, A)^- &= \neg\, A^- \\
(A \to B)^+ &= \neg\,(A^+ \wedge \neg\, B^+) & (A \to B)^- &= \neg\, A^- \vee B^-
\end{aligned}
$$

It is not necessary to have an explicit implication operator in LC; instead we simply unfold the positive and negative interpretations. Notice that $(A^+)^\perp = (A^\perp)^-$ and $(A^-)^\perp = (A^\perp)^+$.

Next, every ZU judgment $\Gamma \vdash J$ will be mapped to a judgment $\vdash [\Gamma]^\perp; \downarrow[J]$ where $[-]$ is defined as follows:

$$
\begin{aligned}
[A\,\mathbf{triv}] &= A^+ \\
[A\,\mathbf{false}] &= (A^\perp)^- \\
[A\,\mathbf{true}] &= A^- \\
[A\,\mathbf{absurd}] &= (A^\perp)^+ \\
[\mathbf{contra}] &= \cdot
\end{aligned}
$$

[4] Zeilberger (2008) defines a translation to a different focused logic, which we extend here to LC.

As expected, direct proofs and refutations are mapped to positive propositions, although the refutations are dualized first. That is, a direct refutation of A is simply a direct proof of $A^\perp$. Similarly, indirect proofs and refutations are negative.

We next show that this translation is in fact sound and complete.

Theorem 2.

1. *(Soundness) If* $\Gamma \vdash J$ *in* ZU *then* $\vdash [\Gamma]^\perp; \downarrow[J]$ *in* LC.
2. *(Completeness) If* $\vdash [\Gamma]^\perp; \downarrow[J]$ *in* LC *then* $\Gamma \vdash J$ *in* ZU.

For soundness, we need a lemma for each judgment except contradiction. For simplicity, we consider only the positive judgments; the others are addressed dually.

Lemma 3. *If* $\Delta \Rightarrow A\,\mathbf{triv}$ *then* $\vdash [\Delta]^\perp; A^+$

The rule for the false judgment is more involved.

Lemma 4. *Suppose that for all* $\Delta_1 \Rightarrow A_1\,\mathbf{triv}, \ldots, \Delta_n \Rightarrow A_n\,\mathbf{triv}$ *we have* $\vdash [\Gamma]^\perp, [\Delta_1]^\perp, \ldots, [\Delta_n]^\perp$. *Then* $[\Gamma]^\perp, ({A_1}^\perp)^-, \ldots, ({A_n}^\perp)^-$.

The proof is by induction on $|A_1| + \cdots + |A_n|$.

The following variation on the cut rule is immediate from the derived $\uparrow$ rule.

Lemma 5. *If* $\vdash \Gamma_1; \downarrow[J]$ *and* $\vdash [J]^\perp, \Gamma_2; \Theta$ *then* $\vdash \Gamma_1, \Gamma_2; \Theta$.

Proof of Theorem 2 (Soundness). The proof is fairly straightforward by induction on the inference rule. In the *triv* judgment case, $\Gamma \vdash A\,\mathbf{triv}$ only if there is some Δ such that $\Gamma \vdash \Delta$ and $\Delta \Rightarrow A\,\mathbf{triv}$. But if $\Gamma \vdash \Delta$ we can write $\Delta = J_1, \ldots, J_n$ and $\Gamma = \Gamma_1, \ldots, \Gamma_n$ such that for each i, $\Gamma_i \vdash J_i$. By induction therefore we have the following judgments: $\vdash [\Gamma_i]^\perp; \downarrow[J_i]$ and $\vdash [J_1]^\perp, \ldots, [J_n]^\perp; A^+$. By Lemma 5 these combine to $\vdash [\Gamma_1]^\perp, \ldots, [\Gamma_n]^\perp; A^+$ as desired.

In the false judgment case, $\Gamma \vdash A\,\mathbf{false}$ provided that for all Δ such that $\Delta \Rightarrow A\,\mathbf{triv}$ we have $\Gamma, \Delta \vdash \mathbf{contra}$. For each such Δ we have $\vdash [\Gamma^\perp], [\Delta^\perp]; \cdot$ by the induction hypothesis, and so by Lemma 4 we know $\vdash [\Gamma^\perp], (A^\perp)^-$. But $\downarrow[A\,\mathbf{false}] = (A^\perp)^- \wedge \mathbf{T}$. We thus obtain the following derivation:

$$\frac{\vdash [\Gamma^\perp], (A^\perp)^-; \cdot \qquad \vdash \cdot; \mathbf{T}}{\vdash [\Gamma^\perp]; (A^\perp)^- \wedge \mathbf{T}}$$

□

The proof of completeness follows from the following lemma:

Lemma 6.

1. *If* $\vdash [\Gamma]^\perp; A^+$ *then* $\Gamma \vdash A\,\mathbf{triv}$.

2. *If* $\vdash [\Gamma]^{\perp}, A^{-}; \cdot$ *then* $\Gamma \vdash A\,\mathbf{true}$.

3. *If* $\vdash [\Gamma]^{\perp}; A^{+}$ *then* $\Gamma \vdash A^{\perp}\,\mathbf{absurd}$.

4. *If* $\vdash [\Gamma]^{\perp}, A^{-}; \cdot$ *then* $\Gamma \vdash A^{\perp}\,\mathbf{absurd}$.

Proof. Items (3) and (4) follow trivially from (1) and (2) considering that for any judgment, $[J] = [J^{\perp}]$. Thus if $\vdash [\Gamma]^{\perp}; A^{+}$ as in item (3), then $\vdash [\Gamma^{\perp}]^{\perp}; A^{+}$ and so by (1) we have $\Gamma^{\perp} \vdash A\,\mathbf{triv}$. By duality in ZU, we know $\Gamma \vdash A^{\perp}\,\mathbf{absurd}$, as expected.

The proof of (1) and (2) follow straightforwardly by induction on the LC derivation. □

Proof of Theorem 2 (Completeness). Consider two cases. If $[J]$ is a positive proposition P, then from $\vdash [\Gamma]^{\perp}; \downarrow P$ we can obtain $\vdash [\Gamma]^{\perp}; P$ from the proof of $\vdash \uparrow P^{\perp}; P$. The result follows from Lemma 6.

Similarly, if $[J]$ is a negative N, then from $\vdash [\Gamma]^{\perp}; \downarrow N$ we can obtain $\vdash [\Gamma^{\perp}], N$ by cutting against the proof $\uparrow N^{\perp}, N \vdash \cdot$. □

It is also worth pointing out the following two completeness results inspired by Zeilberger (2008) relating unfocused classical logic LK to both ZU and LC:

Proposition 7 (Zeilberger, 2008)**.** *If* $\vdash [\Gamma]^{\perp}$ *in LK then* $\Gamma \vdash \mathbf{contra}$ *in ZU.*

Noticed that LK only corresponds to ZU up to *unfocused* judgments.

The corresponding theorem for LC follows naturally:

Proposition 8. *If* $\vdash \Delta$ *is classically provable in LK then* $\vdash \Delta; \cdot$ *has a focusing proof in LC.*

Proof. By induction on the LK derivation. Alternatively, let $(|A|)$ be any expansion of LK types to ZU judgments, in the sense of Danos et al. (1995), meaning that $[(|A|)] = A$ for every LK type A. Then since $\vdash \Delta$ it is also the case that $\vdash [(|\Delta|)]$. By Proposition 7 therefore, $(|\Delta|)^{\perp} \vdash \mathbf{contra}$, so by the soundness of Theorem 2 we have $\vdash [(|\Delta|)]; \cdot$ as a judgment in ZU, as expected. □

5 A Linear Logic Perspective

In Zeilberger's judgmental interpretation, the positive and negative perspectives provide two interpretations of the same connectives. In Girard's work, the polarities partition the connectives into positive and negative fragments. A different perspective of this fragmentation is to treat every connective as having two copies—a positive copy and a negative copy. To distinguish the copies, we use notation from linear logic as summarized in the following chart:

Operator	Positive Copy	Negative Copy
$\mathbf{T}$	1	$\bot$
$\wedge$	$\otimes$	$\parr$
$\mathbf{F}$	0	$\top$
$\vee$	$\oplus$	$\&$
$\neg$	$\overset{+}{\neg}$	$\overset{-}{\neg}$
shift	$\downarrow/!$	$\uparrow/?$

In the spirit of linear logic, Laurent and Regnier use the notation ! and ? in place of the shift operators $\downarrow$ and $\uparrow$ respectively in their presentation of polarized logic, LLP. There are both advantages and disadvantages to their presentation. On the one hand, it draws further attention to the many similarities between linear logic and polarized logic. On the other hand, it confuses the fact that propositions are *not* inherently linear in LLP–they always admit weakening and contraction. We will primarily use the shift notation to remain consistent to the previous presentations, but we will recall the connections to the exponentials often.

The types of LLP are as follows:

$$P ::= 1 \mid P \otimes P \mid 0 \mid P \oplus P \mid \overset{+}{\neg} P \mid \downarrow N$$
$$N ::= \bot \mid N \parr N \mid \top \mid N \,\&\, N \mid \overset{-}{\neg} N \mid \uparrow P$$

The difference between Girard's connectives and Laurent and Regnier's is thus that the positive/negative copies operate exclusively on positive/negative sub-propositions, respectively. In order to change the polarity of a proposition, one of the shift connectives must explicitly be applied.

The inference rules for LLP are given in Figure 6. They are almost identical to Girard's linear logic, with a few structural exceptions. For one, weakening and contraction are applicable on any negative proposition, not just propositions of the form $?\,A$. For another, the $\downarrow$ rule promotes a positive proposition when all the other hypotheses are negative. When $\downarrow$ is equated with !, this again means that all negative propositions are treated like $?\,A$ propositions. In Section 5.2 we will define a translation from LLP into linear logic, and we will need to make this correspondence precise.

Although there is no designated stoup in the judgments of LLP, we do have the following property:

Proposition 9. *In any LLP derivation there is at most one positive proposition.*

$$\frac{}{\vdash P^{\perp}, P} \qquad \frac{\vdash \Delta_1, P \quad \vdash P^{\perp}, \Delta_2}{\vdash \Delta_1, \Delta_2}$$

$$\frac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, \downarrow N} \qquad \frac{\vdash \Delta, P}{\vdash \Delta, \uparrow P}$$

$$\frac{\vdash \Delta}{\vdash N, \Delta} \qquad \frac{\vdash N, N, \Delta}{\vdash N, \Delta}$$

$$\frac{}{\vdash 1} \qquad \frac{\vdash \Delta}{\vdash \bot, \Delta}$$

$$\frac{\vdash \Delta_1, P_1 \quad \vdash \Delta_2, P_2}{\vdash \Delta_1, \Delta_2, P_1 \otimes P_2} \qquad \frac{\vdash N_1, N_2, \Delta}{\vdash N_1 \parr N_2, \Delta}$$

$$\frac{}{\vdash \mathcal{N}, \top}$$

$$\frac{\vdash \Delta, P_i}{\vdash \Delta, P_1 \oplus P_2} \qquad \frac{\vdash N_1, \Delta \quad \vdash N_2, \Delta}{\vdash N_1 \mathbin{\&} N_2, \Delta}$$

$$\frac{\vdash \mathcal{N}, P^{\perp}}{\vdash \mathcal{N}, \overset{+}{\neg} P} \qquad \frac{\vdash \Delta, N^{\perp}}{\vdash \Delta, \overset{-}{\neg} N}$$

Figure 6: LLP (Laurent and Regnier, 2003)

5.1 Translations into LLP

The goal is to make the relationship formal again. Define a translation $\langle A \rangle$ which takes LC propositions to LLP propositions of the same polarity.

$$\begin{aligned}
\langle \mathbf{T} \rangle &= 1 & \langle \overline{\mathbf{T}} \rangle &= \bot \\
\langle \mathbf{F} \rangle &= 0 & \langle \overline{\mathbf{F}} \rangle &= \top \\
\langle P_1 \wedge P_2 \rangle &= \langle P_1 \rangle \otimes \langle P_2 \rangle & \langle P_1 \vee P_2 \rangle &= \langle P_1 \rangle \oplus \langle P_2 \rangle \\
\langle N \wedge P \rangle &= \downarrow\langle N \rangle \otimes \langle P \rangle & \langle N \vee P \rangle &= \langle N \rangle \parr \uparrow\langle P \rangle \\
\langle P \wedge N \rangle &= \langle P \rangle \otimes \downarrow\langle N \rangle & \langle P \vee N \rangle &= \uparrow\langle P \rangle \parr \langle N \rangle \\
\langle N_1 \wedge N_2 \rangle &= \langle N_1 \rangle \mathbin{\&} \langle N_2 \rangle & \langle N_1 \vee N_2 \rangle &= \langle N_1 \rangle \parr \langle N_2 \rangle \\
\langle \neg P \rangle &= \downarrow\langle P \rangle^{\perp} = \,!\,\langle P \rangle^{\perp} & \langle \neg N \rangle &= \uparrow\langle N \rangle^{\perp} = \,?\,\langle N \rangle^{\perp}
\end{aligned}$$

Again this translation is sound and complete.

Proposition 10.

1. *(Soundness) If* $\vdash \Delta; \Theta$ *in LC then* $\vdash \langle \uparrow \Delta \rangle, \langle \Theta \rangle$ *in LLP.*

2. *(Completeness) If* $\vdash \langle \Delta \rangle, \langle \Theta \rangle$ *in* LLP *then* $\vdash \Delta ; \Theta$ *in* LC.

Proof. For soundness, the proof is by induction on the LC derivation.

Completeness can be proved by induction on the LLP derivation, but we can alternatively sketch a more roundabout proof showing the interconnectedness of all the systems we have seen so far. Consider that every proof of LLP can easily be "erased" to a classical (LK) *skeleton* along the lines of Danos et al. (1995). We write this erasure function on propositions to be $|-|$. It is quite easy to see however that for any LC type, $|\langle A \rangle| = A$. Thus the LK proof has the form $\vdash \Delta, \Theta$, and by the completeness of focusing (Proposition 8) there is an LC proof $\vdash \Delta ; \Theta$. $\square$

5.2 Comparing classical and polarized logic

Girard's classification of types into polarities gives one polarization strategy for LK types, but it is not the only one. The simplest strategies translate all types as either positive or negative. Laurent and Regnier (2003) introduce two such translations, which we write here as $\langle - \rangle^+$ and $\langle - \rangle^-$.[5]

In fact, the translations given by Laurent and Regnier factor into Girard's translation plus the positive and negative coercions given in Section 4:

$$\langle A \rangle^+ = \langle A^+ \rangle \qquad \langle A \rangle^- = \langle A^- \rangle$$

The defining feature of the negative translation is that $\langle A \to B \rangle^- = {!}\, \langle A \rangle^- \multimap \langle B \rangle^-$ when we write $!$ for $\downarrow$:

$$\begin{aligned} \langle (A \to B)^- \rangle &= \langle \neg\, A^- \vee B^- \rangle \\ &= \langle \neg\, A^- \rangle \parr \langle B^- \rangle \\ &= {\uparrow}\langle A^- \rangle^{\perp} \parr \langle B^- \rangle \end{aligned}$$

This corresponds to Girard's original translation of intuitionistic logic (LJ) into linear logic (LL) (Girard, 1987). Indeed, Laurent and Regnier (2003) prove that Girard's translation combined with a CPS transformation from LK is the same as the negative translation $\langle - \rangle^-$ composed with a CPS-like transformation from LLP to LL:

$$\begin{tikzcd} \mathsf{LK} \arrow[r, "\text{CPS}"] \arrow[d, "\langle - \rangle^-"'] & \mathsf{LJ} \arrow[d, "{!}\, A \multimap B"] \\ \mathsf{LLP} \arrow[r, "\text{CPS}"'] & \mathsf{LL} \end{tikzcd}$$

Furthermore, they show that this relationship holds for multiple call-by-name CPS translations, for example by Plotkin (1975) and Krivine (2002).

[5]In their paper, Laurent and Regnier (2003) use the plain notation $(-)^+$ and $(-)^-$ for this operation.

Plotkin's translation. Following Laurent and Regnier (2003), we will sketch the translations on the fragment of classical logic restricted to atomic propositions and implication. This fragment types the $\lambda\mu$-calculus (Parigot, 1992), a classically-typed calculus that Laurent and Regnier use to define their translations on proofs. In fact, starting with different sets of atomic primitives restricts the possible expressible translations—for an overview of such CPS translations on the full set of primitives, see Ferreira and Oliva (2011).

Plotkin's translation is based on the following mapping of propositions:

$$X^\bullet := X \qquad (A \to B)^\bullet := \neg\neg A^\bullet \to \neg\neg B^\bullet$$

where $\neg A$ in intuitionistic logic is just $A \to 0$. It is then the case that A is provable classically in LK if and only if $A^\bullet$ is provable intuitionistically in LJ.

In fact, Laurent and Regnier refine the type of $A^\bullet$ with a linear negation type $\neg_0 A$, and Plotkin's translation satisfies $(A \to B)^\bullet := \neg\neg_0 A \to \neg\neg_0 B$. We leave the details to Laurent and Regnier (2003).

Meanwhile, it is necessary to define a map $\mathsf{LLP} \to \mathsf{LL}$ which is compatible with Plotkin's translation in Laurent and Regnier's commuting square. This map is called the *box translation*, written $(-)^{\mathbf{b}}$, and is defined in Figure 7. The box translation adds a bang operator ! around each positive LLP proposition, and adds a why not operator ? around each negative LLP proposition. As expected from Laurent and Regnier's conflation of the two ideas, the shift operators are mapped to the appropriate exponential.

The invariant of having an exponential at the front of $A^{\mathbf{b}}$ for every A means that all negative propositions are explicitly subject to weakening laws and all positive propositions are explicitly duplicable (meaning $P \multimap P \otimes P$ is provable).

Laurent and Regnier show that for any LK type A we have

$$(\langle A \rangle^-)^{\mathbf{b}} = ?\, G\,(A^\bullet)$$

where $G\,(A)$ is Girard's translation $!\, A \multimap B$ from intuitionistic logic LJ to linear logic LL. They also extend the translations to derivations and prove that they too commute.

Krivine's translation. The disadvantage of Plotkin's translation and the corresponding box translation is that they introduce many superfluous negation, ! and ? operators. Both computationally and logically we have motivation to find a more efficient presentation. One solution, due to Krivine (2002), pushes all negations to the atomic propositions:

$$X^* := \neg X \qquad (A \to B)^* := A^* \to B^*$$

It is then the case that A is classically provable in LK if and only if $\neg A^*$ is intuitionistically provable in LJ.

The corresponding translation $\mathsf{LLP} \to \mathsf{LL}$ is called the reversing translation $(-)^\rho$, also shown in Figure 7. In this case, we can still derive the structural rules

$$
\begin{aligned}
X^{\mathbf{b}} &:= !\,X & \overline{X}^{\mathbf{b}} &:= ?\,\overline{X} \\
1^{\mathbf{b}} &:= !\,1 & \bot^{\mathbf{b}} &:= ?\,\bot \\
(P_1 \otimes P_2)^{\mathbf{b}} &:= !\,({P_1}^{\mathbf{b}} \otimes {P_2}^{\mathbf{b}}) & (N_1 \parr N_2)^{\mathbf{b}} &:= ?\,({N_1}^{\mathbf{b}} \parr {N_2}^{\mathbf{b}}) \\
0^{\mathbf{b}} &:= !\,0 & \top^{\mathbf{b}} &:= ?\,\top \\
(P_1 \oplus P_2)^{\mathbf{b}} &:= !\,({P_1}^{\mathbf{b}} \oplus {P_2}^{\mathbf{b}}) & (N_1 \,\&\, N_2)^{\mathbf{b}} &:= ?\,({N_1}^{\mathbf{b}} \,\&\, {N_2}^{\mathbf{b}}) \\
(\downarrow N)^{\mathbf{b}} &:= !\,N^{\mathbf{b}} & (\uparrow P)^{\mathbf{b}} &:= ?\,P^{\mathbf{b}}
\end{aligned}
$$

$$
\begin{aligned}
X^{\rho} &:= !\,X^{\perp} & \overline{X}^{\mathbf{b}} &:= ?\,\overline{X}^{\perp} \\
1^{\rho} &:= 1 & \bot^{\rho} &:= \bot \\
(P_1 \otimes P_2)^{\rho} &:= {P_1}^{\rho} \otimes {P_2}^{\rho} & (N_1 \parr N_2)^{\rho} &:= {N_1}^{\rho} \parr {N_2}^{\rho} \\
0^{\rho} &:= 0 & \top^{\rho} &:= \top \\
(P_1 \oplus P_2)^{\rho} &:= {P_1}^{\rho} \oplus {P_2}^{\rho} & (N_1 \,\&\, N_2)^{\rho} &:= {N_1}^{\rho} \,\&\, {N_2}^{\rho} \\
(\downarrow N)^{\rho} &:= !\,N^{\mathbf{b}} & (\uparrow P)^{\rho} &:= ?\,P^{\rho}
\end{aligned}
$$

Figure 7: The box translation and the reversing translation (Laurent and Regnier, 2003).

for N^{ρ}, but it requires the introduction of a cut against the canonical proof of $?\,N^{\rho} \multimap N^{\rho}$.

Finally, Laurent and Regnier show that $(\langle A \rangle^{-})^{\rho} = G\,(A^{*})$ and similarly for derivations in LK.

Positive translation. Laurent and Regnier (2003) show a similar result for the positive translation $\mathsf{LC} \to \mathsf{LLP}$ satisfying $\langle A \to B \rangle^{+} = !\,(\langle A \rangle^{+} \multimap ?\,\langle B \rangle^{+})$:

$$
\begin{aligned}
\langle A \to B \rangle^{+} &= \langle \neg\,(A^{+} \wedge \neg\, B^{+}) \rangle = {\downarrow}\langle A^{+} \wedge \neg\, B^{+} \rangle^{\perp} \\
&= {\downarrow}(\langle A^{+} \rangle \otimes \langle \neg\, B^{+} \rangle)^{\perp} = {\downarrow}(\langle A^{+} \rangle \otimes {\downarrow}\langle B^{+} \rangle^{\perp})^{\perp} \\
&= {\downarrow}(\langle A^{+} \rangle^{\perp} \parr {\uparrow}\langle B^{+} \rangle)
\end{aligned}
$$

They postulate a similar commuting diagram for call-by-value CPS translations:

$$
\begin{tikzcd}
\mathsf{LK} \arrow[r, "\text{CPS}"] \arrow[d, "\langle - \rangle^{+}"'] & \mathsf{LJ} \arrow[d, "!\,(A \multimap ?\,B)"] \\
\mathsf{LLP} \arrow[r, "\text{CPS}"'] & \mathsf{LL}
\end{tikzcd}
$$

6 Related Work and Conclusion

In this paper, polarization is presented from a proof-theoretic perspective, and as such the emphasis is on the meaning of propositions, judgments, and derivations. Through the Curry-Howard isomorphism, polarization has applications in other domains as well, which we review briefly here.

Evaluation order: CBV vs CBN. Traditionally, evaluation strategies for λ-calculi have been independent from type systems, and so we often have more than one valid evaluation strategy for any particular type system. Zeilberger (2008) points out that as type systems get more precise, however, this property breaks down. One example includes ML value restriction on polymorphism (Milner et al., 1997), which says that only values can have polymorphic type signatures. Without this restriction on types, call-by-name is sound but call-by-value is unsound. Another example arises in a type system with subtyping and intersection types: the standard rule

$$(A \to B) \cap (A \to C) \leq A \to (B \cap C)$$

is sound under call-by-name but unsound under call-by-value.

On the other hand, starting from Levy (2003) and continuing with Curien and Herbelin (2000) and Wadler (2003), it has become clear that the call-by-value and call-by-name disciplines are dual to each other. Polarization solves this incongruity by letting both evaluation disciplines exist in the same type system. In particular, positive types are strict data structures and negative types are lazy data structures; this is the case with sums and products as well as with implication. Polarized type systems with an emphasis on evaluation order have been developed by, for example, Zeilberger (2008), Spiwack (2014), and Stump (2014).

Game semantics and categorical models. In game theory, propositions are interpreted as individual game states, and derivations are interpreted as valid moves on games. Linear duality corresponds to switching the player whose turn it is (proponent versus opponent). Thus, while Hyland and Ong (2000) introduce games that always start with the opponent, Laurent (2004) postulates another collection of games that start with the proponent, and calls these two kinds of games negative and positive respectively. Indeed, these games interpret LLP negative and positive propositions.

Categorical models of polarized logic have also appeared out of the search for game semantics, for example by Cockett and Seely (2007), Hamano and Scott (2007), and Melliès and Tabareau (2010). The basic structure for such polarized categories consists of two categories, one positive and one negative, connected by some adjoint structure.

Focusing versus polarization. The domains of focusing and polarization are closely intertwined, but also stand on their own. Focusing, introduced by

Andreoli (1992) in the context of proof search, makes the observation that right rules for negative connectives are invertible, so may be applied as soon as possible. On the other hand, right rules for positive connectives should be applied as late as possible, but once such a rule is applied, that proposition is considered focused, and right rules can continue to be applied as much as possible.

Although focused logics pay attention to the polarity of a connective, they often do not attribute a notion of polarity to a proposition itself. If anything, the polarity of a proposition may be defined to be the polarity of its outermost connective, in contrast to polarized logic in which the polarity of a proposition depends more on the structure of the proposition.

Although many polarized logics set some kind of focus on a particular proposition—the right of the judgment in ZU, the stoup on LC—the order of operations is not restricted as in Andreoli's focusing system.

Conclusion. In this report we have unpacked the meaning of polarized logic starting from Zeilberger's judgmental meaning-theories. Using this intuition as a starting point, we have considered two other presentations of polarized logic—Girard's original presentation based on classical logic, and Laurent and Regnier's approach based on linear logic. We also compared polarization strategies to double negation translations to show that the intuitionistic understanding of classical logic via CPS translations can be factored into an understanding via polarized logic.

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