A HoTT Quantum Equational Theory

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MURI Project Review
University of Maryland

March 8, 2019

With Steve Zdancewic at the University of Pennsylvania.
Quantum data, classical control

...via embedded languages
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...via embedded languages

- Quipper [Green et al., 2013]
  - Embedded in Haskell, a functional lazy language.
  - Uses Haskell types, functions, data structures, type classes, template haskell... to construct quantum circuits.
  - Access to Haskell REPL and debugging tools.

LiQUiD, Q language, Project Q, QISKit, pyQuill...

- Q wire [Paykin et al., 2017, Rand et al., 2017]
  - A formal theory of embedded quantum circuits.
  - Implemented as an embedded language in Coq, a theorem prover with dependent types.
  - Uses Coq theorem proving capabilities to prove correctness of quantum circuits.
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Quantum/non-quantum calculus

- Based on Linear/Non-Linear (LNL) logic [Benton, 1995]
- Linear types, pairs ($\otimes$), etc
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\[ a : \alpha \]

\[ \text{put } a : \text{QExp} \cdot (\text{Lower } \alpha) \]
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\[
\frac{a : \alpha}{\text{put } a : \text{QExp} \cdot \text{Lower } \alpha}
\]

\[
e : \text{QExp} \Delta \text{ (Lower } \alpha) \quad f : \alpha \rightarrow \text{QExp} \Delta' \tau
\]

\[
e >! f : \text{QExp} (\Delta, \Delta') \tau
\]
Quantum/non-quantum calculus

Derived quantum operations:

- \( \text{Qubit} = \text{Lower} (\text{Bool}) \)
- \( |b\rangle = \text{put} \ b \)
- \( \text{let} \ b := \text{meas} \ e \ \text{in} \ e' = e >! \lambda b.e' \)
Quantum/non-quantum calculus

- Derived quantum operations:
  - \( \text{Qubit} = \text{Lower}(\text{Bool}) \)
  - \( |b\rangle = \text{put } b \)
  - let \( b := \text{meas } e \text{ in } e' = e >! \lambda b.e' \)

- Unitaries (not derived):
  - \( U : U\text{Matrix}(\sigma, \tau) \)
  - \( e : \text{QExp} \Delta \sigma \)
  - \[ U \# e : \text{QExp} \Delta \tau \]
Reasoning about quantum data

- Denotational semantics
  - Spaces are exponential in size of program

Validated with respect to denotational semantics.
Reasoning about quantum data

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- Program logics
  - Best suited to imperative quantum languages
Reasoning about quantum data

- **Denotational semantics**
  - Spaces are exponential in size of program
- **Program logics**
  - Best suited to imperative quantum languages
- **Equational theory**
  - Syntactic rules that characterize when programs are equivalent.
  - May or may not be directed; difficult to normalize.
  - Validated with respect to denotational semantics.
Equational theory for *embedded* quantum circuit language.
Goal

Equational theory for *embedded* quantum circuit language.

- Interaction between quantum data and host language control
- NOT equational theory for classes of unitaries
Prior work – Staton [2015]

- Equational theory for algebra with unitaries and classical control.
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- Equational theory for algebra with unitaries and classical control.

- Complete with respect to $C^*$-algebras.
Prior work – Staton [2015]

- Equational theory for algebra with unitaries and classical control.

- Complete with respect to $C^*$-algebras.
- Procedural axioms based on diagrams
  - symmetric monoidal structure
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- Specialized to an embedded programming language
  - not algebra or diagrams (e.g. ZX calculus [Backens, 2015])
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  - not algebra or diagrams (e.g. ZX calculus [Backens, 2015])
- Fewer “procedural” axioms, focus on interesting axioms.
Equational theory for *embedded* quantum circuit language.

- Specialized to an embedded programming language
  - not algebra or diagrams (e.g. ZX calculus [Backens, 2015])
- Fewer “procedural” axioms, focus on interesting axioms.
- Completeness of axioms by comparing with Staton’s theory.
Homotopy type theory (HoTT): a type theory of equality

- Equality of two terms $a = b$ is a type
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- Terms of equality type $p : a = b$ called *paths*
Homotopy type theory (HoTT): a type theory of equality

- Equality of two terms \( a = b \) is a type
- Constructor: \( 1_a : a = a \)
- Terms of equality type \( p : a = b \) called *paths*
- Path induction:

\[
H : \forall (a, b : A). \ a = b \to \text{Type} \quad \forall (a : A). \ H(1_a)
\]

\[
\text{path\_ind}_H : \forall (a, b : A). \ \forall (p : a = b). \ H(p)
\]
Equality of two terms $a = b$ is a type

Constructor: $1_a : a = a$

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Path induction:

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\]

Equivalence class of an element $a : A$ with respect to a relation $R$: $[a]_R = [b]_R$ if $(a, b) \in R$. 

Higher Inductive Type (HIT)

Definition
The quotient of a type $A$ by a relation $R : A \to A \to \text{Prop}$ is a type $A/R$ with data constructor:

$$a : A \quad [a]_R : A/R$$

...
**Higher Inductive Type (HIT)**

**Definition**
The quotient of a type $A$ by a relation $R : A \to A \to \text{Prop}$ is a type $A/R$ with data constructor:

$$
\frac{a : A}{[a]_R : A/R}
$$

... and path constructor:

$$
\frac{a, b : A \quad p : R(a, b)}{[p] : [a]_R = [b]_R}
$$
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\begin{align*}
  a : A \\
  [a]_R : A/R
\end{align*}
$$

... and path constructor:

$$
\begin{align*}
  a, b : A & \quad p : R(a, b) \\
  [p] : [a]_R = [b]_R
\end{align*}
$$

Note
If $p, q : R(a, b)$ and $p \neq q$, then $[p] \neq [q]$. 
So what?

- HITs use paths to represent *equivalence relations* or *groupoids*. 
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  - Prove theorems about groupoids by showing property holds of $1_a : a = a$. 
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- HITs use paths to represent *equivalence relations or groupoids*.
- Path induction still holds of HITs:
  - Prove theorems about groupoids by showing property holds of $1_a : a = a$.
- Unitary transformations form a groupoid.
Idea: Represent Unitaries as paths

$\text{UMatrix}(\alpha, \beta)$ is the type of unitary matrices of dimension $|\alpha| \times |\beta|$.

$\alpha, \beta$: FinType

Quantum types: QType = FinType / \text{UMatrix}.

Qubit = [Bool] \text{UMatrix}

Unitaries are paths: $U \in \text{UMatrix}(\alpha, \beta)$; $U \in [\alpha] = [\beta]$

$H$: Qubit = Qubit
Idea: Represent Unitaries as paths

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Idea: Represent Unitaries as paths

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  - $\alpha, \beta : \text{FinType}$
- Quantum types: $\text{QType} = \text{FinType}/\text{UMatrix}$.
  - $\text{Qubit} = [\text{Bool}]_{\text{UMatrix}}$
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- Unitaries are paths:

\[
\frac{U : \text{UMatrix}(\alpha, \beta)}{[U] : [\alpha] = [\beta]}
\]

- $[H] : \text{Qubit} = \text{Qubit}$
\begin{align*}
\sigma & \in \text{QType} = \text{FinType/UMatrix} \\
\text{Lower } \alpha & \equiv [\alpha]_{\text{UMatrix}} \\
e & \equiv x \mid \text{let } x := e \text{ in } e' \\
& \mid (e_1, e_2) \mid \text{let } (x_1, x_2) := e \text{ in } e' \\
& \mid \text{put } a \mid e >! f \mid \cdots
\end{align*}
σ ∈ QType = FinType/UMatrix

Lower α ≡ [α]_{UMatrix}

e := x | let x := e in e'

| (e_1, e_2) | let (x_1, x_2) := e in e'

| put a | e >! f | ···

- Derive |b⟩ and meas e using Lower
HoTT QNQ calculus

\[ \sigma \in \text{QType} = \text{FinType}/\text{UMatrix} \]

Lower \( \alpha \equiv [\alpha]_{\text{UMatrix}} \)

\[ e := x \mid \text{let } x := e \text{ in } e' \]

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\[ \mid \text{put } a \mid e >! f \mid \cdots \]

- Derive \( |b\rangle \) and meas \( e \) using Lower
- Derive unitaries...
Theorem

Let $U$ be a unitary transformation $U : \sigma = \tau$.
$(\sigma, \tau : QType \equiv FinType/UMatrix)$

If $\Delta \vdash e : \sigma$ then there exists another expression $\Delta \vdash U \# e : \tau$.
(apply the unitary $U$ to $e$)
Unitaries in HoTT QNQ

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Proof.
By path induction. The proposition is true for $1_\sigma : \sigma = \sigma$:

$$1_\sigma \# e \equiv e$$
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Theorem

Let $U$ be a unitary transformation $U : \sigma = \tau$.

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Note

$[H] \# e \neq e$ because $[H] \neq 1_{Qubit}$
Unitaries in the HoTT QNQ

Theorem

Let $U : \sigma = \tau$ and $V : \tau = \rho$ be unitary transformations. Then

$$V \# (U \# e) = (V \circ U) \# e.$$
Unitaries in the HoTT QNQ

**Theorem**

Let $U : \sigma = \tau$ and $V : \tau = \rho$ be unitary transformations. Then

$$V \# (U \# e) = (V \circ U) \# e.$$ 

**Proof.**

By path induction on $V$. If $V \equiv 1_{\tau}$ then

$$LHS = 1_{\tau} \# (U \# e) = U \# e$$
$$RHS = (1_t \circ U) \# e = U \# e$$
We can prove a lot...

Theorem
\[ U^\dagger \# (U \# e) = e \]
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Theorem
\[ (U_1 \otimes U_2) \# (e_1, e_2) = (U_1 \# e_1, U_2 \# e_2) \]
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\[ X \# |0\rangle = |1\rangle \quad \text{meas}(X \# e) = \neg \text{meas}(e) \]
...but not everything

Theorem

\[ \text{SWAP} \# (e_1, e_2) = (e_2, e_1) \]

Proof.

????
...but not everything

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Proof.

????

Theorem

let \((x, y) := \text{SWAP} \ # e in e' = let (y, x) := e in e'\]

Proof.

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...but not everything

Theorem

\[ \text{SWAP} \ # (e_1, e_2) = (e_2, e_1) \]

Proof.

Similar results for behavior of other “structural” unitaries:

\[ \text{ASSOC} : \sigma_1 \otimes (\sigma_2 \otimes \sigma_3) = (\sigma_1 \otimes \sigma_2) \otimes \sigma_3 \]

\[ \text{LUNIT} : () \otimes \sigma = \sigma \]

\[ \vdots \]
Partial initialization axiom

SWAP is a structural equivalence of type $\forall X, Y. X \otimes Y \rightarrow Y \otimes X$ defined by the function

$$\text{swap}(x, y) = (y, x)$$
Partial initialization axiom

SWAP is a structural equivalence of type $\forall X, Y. \; X \otimes Y \rightarrow Y \otimes X$
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Structural equivalences all correspond to unitaries

$$\hat{\text{swap}} : \forall \sigma, \tau. \; \sigma \otimes \tau = \tau \otimes \sigma$$
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The partial initialization a state $X \otimes Y$ is a pair of expressions.

$$\text{init}_X e \equiv e$$

$$\text{init}_{\text{Qubit}} (b : \text{Bool}) \equiv |b\rangle$$

$$\text{init}_{\sigma \otimes \tau} (a, b) \equiv (\text{init}_\sigma a, \text{init}_\tau b)$$
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Axiom

Let $f$ be a structural equivalence. Then

$$\hat{f} \# \text{init}(b) \approx \text{init}(f(b))$$
Partial measurement axiom

*Partial measurement or partial observation:*

\[
\text{match}_X \ e \ \text{with} \ f = \text{let} \ x := e \ \text{in} \ f \ x
\]

\[
\text{match}_{\text{Qubit}} \ e \ \text{with} \ f = e >! f
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\[
\text{match}_{\sigma \otimes \tau} \ e \ \text{with} \ f = \text{let} \ (x, y) := e \ \text{in}
\]

\[
\text{match}_{\sigma} x \ \text{with} \ (\text{match}_{\tau} y \ \text{with} \ f(x, y))
\]

\[
\ldots
\]
Partial measurement axiom

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\[
\text{match}_\sigma x \text{ with } (\text{match}_\tau y \text{ with } f(x, y))
\]
\[
\ldots
\]

Axiom

Let \( f \) be a structural equivalence. Then:

\[
\text{match} \ \hat{f} \neq e \text{ with } g \simeq \text{match} \ e \text{ with } g \circ f
\]
Results

- Two axioms:
  - structural unitaries + initialization
  - structural unitaries + measurement
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- Quantum programming language embedded in HoTT
  - (Finite) classical data, tuples, and sums
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- Quantum programming language embedded in HoTT
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- Complete with respect to Staton’s equational theory

Sound with respect to density matrices
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  - structural unitaries + measurement
- Quantum programming language embedded in HoTT
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- Complete with respect to Staton’s equational theory
- Sound with respect to density matrices
Results

Pros: theorems for free with path induction

Cons:
- theorems not actually free
- no normalization
- steep learning curve

Takeaway: Equations stem (mostly) from quantum data/classical control, not artificial axioms

Thanks!
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Questions?

Supported by FA9550-16-1-0082
Semantics and Structures for Higher-level Quantum Programming Languages
References


